Outline

- Field Theory in Curved Spacetime
- The Graviton
- Gauge Connection
- Topological Gravity

FT in curved spacetime

Flat space $\longrightarrow$ Curved Space

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$$

invariant space time interval becomes:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

taking, $x \rightarrow x'(x)$

$$ds^2 = g'_{\lambda\sigma}\, dx'^\lambda dx'^\sigma$$

$$= g'_{\lambda\sigma} \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} dx^\mu dx^\nu$$

$$\Rightarrow \quad g_{\mu\nu} = g'_{\lambda\sigma} \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu}$$

Lagrangian is Lorentz invariant

but action integral transforms by

$$d^4x \to d^4x' = d^4x \det\left(\frac{\partial x'}{\partial x}\right)$$

since $$g \equiv \det g_{\mu\nu} = \det\left(g'_{\lambda\sigma} \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}}\right)$$

$$= g'\left[\det\left(\frac{\partial x'}{\partial x}\right)\right]^2$$

$$\Rightarrow \quad d^4x \sqrt{-g} = d^4x \det\left(\frac{\partial x'}{\partial x}\right)\sqrt{-g'}$$

$$= d^4x' \sqrt{-g'}$$

$$\Rightarrow \quad S = \int d^4x \sqrt{-g}\, \mathcal{L}(x, \varphi, g)$$

is invariant under coordinate transformations

Quantizing Gravity: The Graviton

Naively we could write the action for the universe as

$$S = S_g + S_M$$

and quantize gravity by calculating

$$\int \mathcal{D}g\, \mathcal{D}\varphi \; e^{iS}$$

But this is not renormalizable and is not easy to work with!

Einstein showed us that gravity is linked to energy/momentum

the graviton is then the particle associated with the field $g_{\mu\nu}$.

defining the stress energy tensor

$$T^{\mu\nu}(x) = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}(x)}$$

and expanding around flat spacetime by writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\Rightarrow S_M(h) = S_M(h=0) + \int d^4x \sqrt{-g}\, h_{\mu\nu} \frac{\delta S_M}{\delta g_{\mu\nu}}$$

$$= S_M - \int d^4x \, \frac{1}{2} h_{\mu\nu} T^{\mu\nu} + O(h^2)$$

$$g = \det(g_{\mu\nu}) = \det(\eta_{\mu\nu} + h_{\mu\nu})$$

$$= -\det(1 + \eta^{\mu\nu} h_{\mu\nu})$$

$$= -\left(1 + \eta^{\mu\nu} h_{\mu\nu} + \mathcal{O}(h^2)\right)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2)$$

The Einstein-Hilbert Action

$$S \equiv \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R$$

$$= M_p^2 \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$$

Ricci Tensor

The Weak Field Action

taking $x^\mu \to x^\mu - \varepsilon^\mu(x)$

$$\Rightarrow \quad h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

* Similar to gauge transformation of

$$A_\mu \to A_\mu - \partial_\mu \Lambda$$

Since S must be invariant under coordinate transforms and if we expand out to $O(h^2)$ with

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu} h + O(h^2)$$

leads to

$$S_{wfg} = \int d^4x \left(\frac{1}{2} M_p^2 \, \mathcal{I} - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right)$$

where

$$\mathcal{I} = \frac{1}{2} \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu$$
$$- \partial_\lambda h^{\lambda\nu} \partial^\mu h_{\mu\nu} + \partial^\nu h^\lambda_\lambda \partial^\mu h_{\mu\nu}$$

The Graviton Propagator

trick 1 : add $\left(\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h^\lambda_\lambda \right)^2$

to action

$$\Rightarrow S_{wfg} = \int d^4x \, \frac{1}{2} \left[\frac{M_p^2}{2} \left(\partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \frac{1}{2} \partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu \right) - h_{\mu\nu} T^{\mu\nu} \right]$$

this is also known as <u>Harmonic Gauge</u>

as we are free to choose $h_{\mu\nu}$ s.t. $\boxed{\partial^{\mu} h_{\mu\nu} = \frac{1}{2} \partial_{\nu} h^{\lambda}_{\lambda}}$

we can then write

$$S = \frac{-1}{2} M_p^2 \int d^4x \left[h^{\mu\nu} K_{\mu\nu;\lambda\sigma} \Box h^{\lambda\sigma} + \mathcal{O}(h^3) \right]$$

with $K_{\mu\nu;\lambda\sigma} = \frac{1}{2} \left(\eta_{\mu\lambda} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\lambda} - \eta_{\mu\nu} \eta_{\lambda\sigma} \right)$

because $h_{\mu\nu}$ is symmetric, in harmonic gauge

$K^{-1} = K$! so

$$D_{\mu\nu,\lambda\sigma}(k) = \frac{K_{\mu\nu;\lambda\sigma}}{k^2 + i\varepsilon}$$

Differential Geometry

a simple <u>k-form</u>:

$$\varphi = \varphi_{\mu_1 \mu_2 \dots \mu_k} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_k}$$

$$d\varphi = \frac{1}{k!} \partial_\nu \varphi_{\mu_1 \mu_2 \dots \mu_k} dx^\nu dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_k}$$

$* \quad dd = 0 \quad *$

for a locally Euclidean Riemannian Manifold

$$g_{\mu\nu}(x) = e^a_\mu \delta_{ab} e^b_\nu$$

(a similarity transform of sorts)

$a = 1, \dots, D$

"vielbeins" or "world vectors"

↳ many legs

↳ "square root of metric"

On a curved manifold, parallel transport infinitesimally rotates the vielbein:

$$de^a = -\omega^{ab} e^b$$

also

$$e^a = e^a_\mu dx^\mu$$

vielbeins transform by rotations:

$$e^a(x) \to e'^a(x) = O^a_b(x)\, e^b(x)$$

and $\omega \to \omega'$

defined by $de'^a = -\omega'^{ab} e^b$

An instructive calculation:

$$de'^a = d(O^a_b e^b)$$

$$= d(O^a_b e^b_\mu dx^\mu)$$

$$= \partial_\nu (O^a_b e^b_\mu) dx^\nu dx^\mu$$

$$= \partial_\nu O^a_b dx^\nu e^b + O^a_b \partial_\nu e^b_\mu dx^\nu dx^\mu$$

$$= dO^a_b e^b + O^a_b de^b$$

* Liebniz Property! *

$$= dO^a_b e^b - O^a_b \omega^{bc} e^c$$

$$= (dO^a_b O^b_d - O^a_b \omega^{bc} O^c_d) e'^d$$

$$\Rightarrow \omega' = O \omega O^T - (dO) O^T$$

the 1-form ω transforms just like the non-abelian gauge potential A^a_μ

But (!), no analog of e ...

ω is the <u>connection</u> between local Euclidean frames, and varies in a curved manifold!

$$\boxed{d\omega + \omega^2 = R}$$

& for completeness

$$\boxed{de + \omega e = 0}$$

just like $F = dA + A^2$ field strength in NAGT

In Einstein Gravity in order for fields and their derivatives to transform in the same way we must introduce a covariant derivative, D to act on a vector field W:

$$D_\lambda W^\mu = \partial_\lambda W^\mu + \Gamma^\mu_{\lambda\nu} W^\nu$$

Christoffel Symbol

* for spin $\frac{1}{2}$ fields *

$$\mathcal{L} = \bar{\psi}\left(i\gamma^a \eta_{ab} e^{b\mu} D_\mu - m\right)\psi$$

where $D_\mu = \partial_\mu - \frac{i}{4}\omega_{\mu ab}\sigma^{ab}$

because ψ is defined in local Lorentz frame (spinors, not 4-vectors?)

Topological Field Theory

in $(2+1)$-D spacetime

anyons emerge, which aquire a phase $e^{i\frac{\Theta}{\pi}\Delta\varphi}$ after being rotated $\Delta\varphi$ around each other CC

this can be accounted for by with

$$\mathcal{L} = \mathcal{L}_0 + \underbrace{\gamma\,\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda}_{\text{Chern-Simons term}} + a_\mu j^\mu$$

$$a_\mu \to a_\mu + \partial_\mu \Lambda$$

$$\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \to \varepsilon^{\mu\nu\lambda} (a_\mu + \partial_\mu \Lambda) \partial_\nu (a_\lambda + \partial_\lambda \Lambda)$$

$$= \varepsilon^{\mu\nu\lambda} \left(a_\mu \partial_\nu a_\lambda + \partial_\mu \Lambda \partial_\nu a_\lambda \right.$$
$$\left. a_\mu \partial_\nu \partial_\lambda \Lambda + \partial_\mu \Lambda \, \partial_\nu \partial_\lambda \Lambda \right)$$

$$= \varepsilon^{\mu\nu\lambda} \left(a_\mu \partial_\nu a_\lambda + \partial_\mu \Lambda \, \partial_\nu a_\lambda \right)$$

$$\delta S = \gamma \int d^3x \, \varepsilon^{\mu\nu\lambda} \partial_\mu (\Lambda \partial_\nu a_\lambda)$$

$\hookrightarrow$ Dropped

$\Rightarrow$ CS Gauge invariant

CS action

$$S = \gamma \int_M d^3x \; \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

$$= \gamma \int_M d^3x \; a \wedge da$$

now under coordinate transformations

a vector:

$$a_\mu(x) = \frac{\partial x'^\lambda}{\partial x^\mu} a'_\lambda(x')$$

so

$$\varepsilon^{\mu\nu\lambda} a_\mu(x) \, b_\nu(x) \, c_\lambda(x)$$

$$= \varepsilon^{\mu\nu\lambda} \frac{\partial x'^\sigma}{\partial x^\mu} \frac{\partial x'^\tau}{\partial x^\nu} \frac{\partial x'^\rho}{\partial x^\lambda} a_\sigma(x') \ldots$$

$$= \det\left(\frac{\partial x'}{\partial x}\right) \varepsilon^{\sigma\tau\rho} a'_\sigma(x') \, b'_\tau(x') \, c'_\rho(x')$$

$$\Rightarrow \quad d^3x \, \varepsilon^{\mu\nu\lambda} a_\mu(x) \, b_\nu(x) \, c_\lambda(x)$$

$$= d^3x' \, \varepsilon^{\sigma\tau\rho} a'_\sigma(x') \, b'_\tau(x') \, c'_\rho(x')$$

with $\quad d^3x' = d^3x \det\left(\frac{\partial x'}{\partial x}\right)$

$\Rightarrow$ no metric,

$$\int Da \; e^{i\int_M a\, da}$$

↳ closed manifold

depends only on <u>topology</u>

but what about $T^{\mu\nu}$

no $g_{\mu\nu} \Rightarrow T^{\mu\nu} = 0$

$\Rightarrow H = 0$

$\Rightarrow$ Ground state degeneracy!

CS Gravity

in terms of connection ω

$$R_{ij}{}^{a}{}_{b} = \partial_i \omega_j{}^{a}{}_{b} - \partial_j \omega_i{}^{a}{}_{b}$$

$$+ [\omega_i, \omega_j]^{a}{}_{b}$$

or

$$R = d\omega + \omega \wedge \omega$$

and in $d = 4$

$$S_{EH} = \frac{1}{2} \int_M \varepsilon^{ijkl} \epsilon_{abcd} \left(e_i^a e_j^b R_{kl}^{cd} \right)$$

$$= \frac{1}{2} \int_M e \wedge e \wedge R$$

$$= \frac{1}{2} \int_M e \wedge e \wedge (d\omega + \omega^2)$$

ω has group structure $SO(3,1)$

interpreting e's as translations

(e, ω) is in $ISO(3,1)$

↳ GR is ISO Gauge theory?

not in $d = 4$

no such gauge action of

form

$$\int A \wedge A \wedge (dA + A^2)$$

in $(2+1)-D$:

$$S_{EH} = \frac{1}{2} \int_M e \wedge (d\omega + \omega^2)$$

$$\downarrow$$

looks like CS

$$S_{CS} = \frac{1}{2} \int_M \mathrm{Tr}\left(A\, dA + \frac{2}{3} A^3\right)$$

A a Lie-algebra valued 1-form

"Tr" a non-degenerate invariant

bilinear form on Lie Algebra

$$A = A^a T_a \Rightarrow \mathrm{Tr}(A\, dA) = \underbrace{\mathrm{Tr}(T_a T_b)}_{d_{ab}} \times A^a \wedge dA^b$$

"metric"

$ISO(2,1)$ Gauge Theory:

$$[J_a, J_b] = \varepsilon_{abc} J^c$$
$$[J_a, P_b] = \varepsilon_{abc} P^c$$
$$[P_a, P_b] = 0$$

J^{ab} Lorentz generators

P^a translation generators

let $J^a = \frac{1}{2} \varepsilon^{abc} J_{bc}$

Then the gauge field is the 1-form

$$A_i = e_i^a P_a + \omega_i^a J_a$$

$$A = Pe + J\omega$$

an infinitesimal gauge parameter would be

$$u = \rho^a P_a + \tau^a J_a$$

$$\Rightarrow \quad \delta A_i = - D_i u$$

with $\quad D_i u = \partial_i u + [A_i, u]$

$$D = d + A$$

$$\Rightarrow \quad D^2 = (d + A)(d + A)$$

$$= d^2 + A d + dA + A^2$$

$$= (dA) + A^2$$

$\hookrightarrow$ function!

$$(dA + Ad)\,\omega = d(A\omega) - A\,d\omega$$

$$= (dA)\,\omega - 2A\,d\omega$$

$$\Rightarrow \quad Ad = (dA) - Ad$$

$$F = D^2$$

$$= (dA) + A^2$$

$$= d(Pe + J\omega) + (Pe + J\omega)^2$$

$$= P\,de + J\,d\omega + P(e\omega + \omega e)$$

$$+ J\omega^2$$

$$= P(de + \omega e) + J(d\omega + \omega^2)$$

$$= 0 + J\;R$$

$$F \propto R$$

$$S_{cs} = \frac{1}{2}\int d^3x \; \mathrm{Tr}\left(A \wedge D^2\right)$$

$$\propto \frac{1}{2}\int d^3x \; A \wedge R$$

$$\propto \underbrace{\frac{1}{2}\int d^3x \; e \wedge R}_{S_{EH}} \; + \; S_\omega$$

for constant curvature:

$$\Rightarrow \int De\,D\omega \; e^{iS_{cs}}$$

$$= \left(\int D\omega \, e^{iS_\omega}\right)\left(\int De \, e^{iS_{EH}}\right)$$